

Power-Law Distributions: Beyond Paretian Fractality

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Abstract

The notion of fractality, in the context of positive-valued probability distributions, is conventionally associated with the class of Paretian probability laws. In this research we show that the Paretian class is merely one out of six classes of probability laws – all equally entitled to be ordained fractal, all possessing a characteristic power-law structure, and all being the unique fixed points of renormalizations acting on the space of positive-valued probability distributions. These six fractal classes are further shown to be one-dimensional functional projections of underlying fractal Poisson processes governed by: **(i)** a common elemental power-law structure; and, **(ii)** an intrinsic scale which can be either linear, harmonic, log-linear, or log-harmonic. This research provides a panoramic and comprehensive view of fractal distributions, backed by a unified theory of their underlying Poissonian fractals.

Keywords: Paretian fractality; renormalization; Poisson processes; Poissonian fractality and renormalization; Fréchet, Weibull, and Lévy Stable distributions.

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1 Introduction

Fractal objects are ubiquitous across many fields of science, and their study has attracted major interest by a broad array of researchers – see [1]-[4] and references therein. The geometric characteristic of fractals is *invariance under changes of scale*. The algebraic manifestation of scale-invariance is given by *power-laws*.

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Power-laws facilitate the characterization of fractality when no geometry is present – the quintessential example being power-law probability distributions (see Chapter 38 in [1]).

Consider a population represented by a collection of points scattered arbitrarily on the positive half-line – the points representing the values of the population members. Examples include: earthquakes taking place in a given geological region, during a given period of time, measured by their magnitudes – each point representing the magnitude of an earthquake; stars in a given sector of space measured by their masses – each point representing the mass of a star; citizens of a given state measured by their wealth – each point representing the wealth of a citizen; insurance claims in a given insurance-portfolio measured by their costs – each point representing the cost of a claim; etc.

Such populations are discrete objects possessing no natural geometry – and hence no natural geometric characterization of fractality. The natural setting for the analysis of such populations is statistical – providing the following conventional algebraic-statistic definition of fractality: *a population is fractal if its population-values and their occurrence-frequencies are connected via a power-law.*

Shifting from the statistical perspective to the probabilistic perspective one picks at random a member of the population, and considers its random value X . Fractality, in the probabilistic setting, is characterized by a *power-law survival probability* of the random variable X :

$$\mathbf{Prob}\left(X > x\right) = \left(\frac{a}{x}\right)^\alpha \quad (1)$$

($x > a$); the parameter a being an arbitrary positive lower bound, and the parameter α being an arbitrary positive exponent.

The probability distribution corresponding to the survival probability of equation (1) is referred to as *Paretian* – named after the Italian economist Vilfredo Pareto who discovered, in 1896, a power-law distribution of wealth in human societies [5]. The Paretian probability distribution was empirically observed in a multitude of examples coming from diverse scientific fields [6]-[9] (see also the review [10] and references therein).

The theoretical construction of the Paretian power-law probability distribution is based on the following pair of foundations: **(i)** fractals are characterized, algebraically, by power-laws; **(ii)** probability distributions are characterized, statistically, by survival probabilities.

The first foundation implicitly assumes that “fractality” is synonymous with “power-laws”. This implicit assumption is *false*. The notion of fractality – in the case of populations represented by arbitrarily-scattered real-valued points – can be defined from *first principles*. Namely, fractality can be defined via the elemental geometric notion of *scale-invariance* – rather than via the emergent algebraic notion of *power-laws*. This approach, undertaken in [11], yields three classes of *non-Paretian* fractal populations.

The second foundation implicitly assumes that random variables are *uniquely* characterized by their survival probabilities. This implicit assumption is, again, false. Indeed, there are many ways of characterizing a given probability distribution (these characteristics will be rigorously defined in the sequel): Cumulative Distribution Functions (CDFs); Survival Distribution Functions (SDFs); Backward Hazard Rates (BHRs); Forward Hazard Rates (FHRs); Laplace Transforms (LTs); Moment Sequences (MSs); Log-Laplace Transforms (LLTs); Cumulant Sequences (CSs).

Associating “fractal distributions” with power-law survival probabilities yields Paretian probability distributions. But what if we associate “fractal distributions” with power-law Hazard Rates? or with power-law Log-Laplace transforms? or with power-law Cumulants? This question serves as the starting point of our research.

This paper is devoted to the exploration of the *definition of fractality* in the context of positive-valued probability distributions. As we shall demonstrate, the notion of fractality is highly contingent on the distribution-characteristic used. Defining fractality via power-law structures of different distribution-characteristic leads to markedly different probability distributions including: *Pareto*, *Beta*, *Fréchet*, *Weibull*, *Lévy Stable* – all equally entitled to be considered “fractal distributions”.

Altogether we characterize six different classes of fractal distributions – each class emerging from a power-law structure of a different distribution-characteristic. Each of the six fractal classes characterized is shown to be associated with a different *renormalization*: the members of each fractal class are the *unique fixed points* of a specific renormalization acting on the space of positive-valued probability distributions. Each of the six fractal classes characterized is also associated with a different *Poissonian representation*: the members of each fractal class are representable as a *functional projection* of an underlying class of Poisson processes defined on the positive half-line.

Having characterized the six different classes of fractal distributions, their renormalizations, and their Poissonian representations, we turn to seek an underlying *unifying fractal structure*. To that end we study *Poissonian renormalizations* – renormalizations of Poisson processes defined on the positive half-line – and characterize four classes of *Poissonian fractals*: the unique fixed points of multiplicative and power-law Poissonian renormalizations.

The Poissonian fractals turn out to be governed by two structures: **(i)** a *power-law* structure common to all Poissonian fractal classes; **(ii)** an *intrinsic scale* which differentiates between the four Poissonian fractal classes and characterizes them. The intrinsic scale can be either linear, harmonic, log-linear, or log-harmonic. Moreover, the Poissonian fractals turn out to be the Poisson processes underlying the aforementioned fractal distributions. And, the “algebraic fractality” on the “probability-distribution level” turns out to be a one-dimensional projection of a more elemental “geometric fractality” prevalent on the underlying “Poisson-process level”.

This research provides a panoramic and comprehensive view of fractal distributions, backed by a unified theory of their underlying Poissonian fractals. The manuscript is organized as follows. The six classes of fractal distributions – as well as their associated renormalizations – are characterized in Section 2. Poissonian representations of the fractal distributions are presented in Section 3. The Poissonian renormalizations and Poissonian fractals underlying the six classes of fractal distributions are unveiled in Section 4.

Acronym glossary

Throughout the manuscript the following acronyms shall be frequently used (the subsections in brackets indicate the location, in the manuscript, of the corresponding definitions):

IID = Independent and Identically Distributed

PDF = Probability Density Function (Subsection 2.1)

CDF = Cumulative Distribution Function (Subsection 2.1)

SDF = Survival Distribution Function (Subsection 2.1)

BHR = Backward Hazard Rate (Subsection 2.2)

FHR = Forward Hazard Rate (Subsection 2.2)

LT = Laplace Transform (Subsection 2.3)

MS = Moment Sequence (Subsection 2.3)

LLT = Log-Laplace Transform (Subsection 2.3)

CS = Cumulant Sequence (Subsection 2.3)

CRF = Cumulative Rate Function (Subsection 3.1)

SRF = Survival Rate Function (Subsection 3.1)

The equality sign $\stackrel{\text{Law}}{=}$ shall henceforth denote equality in law (of random variables).

2 Power-law characterization of fractal distributions

As noted in the introduction, positive-valued probability distributions have various distribution-characteristics. In general, a distribution-characteristic \mathcal{C}_D of a positive-valued probability distribution D is a function $\mathcal{C}_D = \mathcal{C}_D(\theta)$ ($\theta \in \Theta$; Θ being a subset of the non-negative half line) which uniquely determines D .

The aim of this research is to explore the notion of *fractality*, in the context of positive-valued probability distributions, via the following definition:

Definition 1 A probability distribution D is \mathcal{C} -fractal if its distribution-characteristic \mathcal{C}_D admits a power-law functional structure:

$$\mathcal{C}_D(\theta) = c\theta^\gamma \quad (2)$$

($\theta \in \Theta$), where c is a positive coefficient and where γ is a real exponent.

Often, fractality is the manifestation of some underlying *renormalization*. In the context of positive-valued probability distributions a renormalization \mathcal{R} is a family of transformations $\mathcal{R} = \{\mathcal{R}_p\}_{p>0}$ – mapping probability distributions to probability distributions – which is *consistent*: A p -renormalization followed by a q -renormalization equals a pq -renormalization: $\mathcal{R}_p \circ \mathcal{R}_q = \mathcal{R}_{pq}$ ($p, q > 0$; the sign \circ denoting composition).

A probability distribution D is a *fixed point* of the renormalization \mathcal{R} if it is a fixed point of each of the renormalization's transformations: $\mathcal{R}_p(D) = D$ (for all $p > 0$). The connection between \mathcal{C} -fractal probability distributions and renormalizations is given by the following definition:

Definition 2 A renormalization \mathcal{R} is \mathcal{C} -fractal if its set of fixed points coincides with the set of \mathcal{C} -fractal probability distributions.

In this section we study \mathcal{C} -fractality with regard to each of the distribution-characteristics specified above. As shall be demonstrated, different distribution-characteristics will lead to very different meanings of fractality.

2.1 Fractality via Frequencies

Pareto's approach to analyzing the empirical data he gathered was based on *frequencies*: studying the occurrence-frequencies of the different population-values. In other words, Pareto focused on the Probability Density Function (PDF) $f_D(\cdot)$ of a given probability distribution D .

The PDF $f_D(\cdot)$, in turn, induces the two most fundamental distribution-characteristics of a probability distribution D : **(i)** the Cumulative Distribution Function (CDF) $F_D(\cdot)$, given by

$$F_D(\theta) = \int_0^\theta f_D(x)dx \quad (3)$$

($\theta > 0$); and, **(ii)** the Survival Distribution Function (SDF) $\overline{F}_D(\cdot)$, given by

$$\overline{F}_D(\theta) = \int_\theta^\infty f(x)dx \quad (4)$$

($\theta > 0$).

In this Subsection we study CDF-fractality and SDF-fractality.

2.1.1 CDF-fractality

The CDF $F_D(\cdot)$ is monotone increasing from the level $\lim_{\theta \rightarrow 0} F_D(\theta) = 0$ to the level $\lim_{\theta \rightarrow \infty} F_D(\theta) = 1$. Hence, in order that the CDF $F_D(\cdot)$ admit a power-law structure its underlying probability distribution D must be *bounded from above*. Admissible power-law CDFs are thus of the form

$$F_D(\theta) = \left(\frac{\theta}{a}\right)^\alpha \quad (5)$$

($0 < \theta < a$), where the upper bound a and the exponent α are arbitrary positive parameters.

With no loss of generality, the upper bound can be set to unity ($a = 1$) – yielding the *Beta* CDFs:

$$F_D(\theta) = \theta^\alpha \quad (6)$$

($0 < \theta < 1$).

Let ξ denote a random variable drawn from an arbitrary probability distribution D supported on the unit interval $(0, 1)$. The *conditional* distribution of the *scaled* random variable ξ/p – contingent on the information that the random variable ξ is *no greater* than the level p – is given by

$$\mathbf{Prob}\left(\frac{\xi}{p} \leq \theta \mid \xi \leq p\right) = \frac{\mathbf{Prob}(\xi \leq p\theta)}{\mathbf{Prob}(\xi \leq p)} \quad (7)$$

($0 < p, \theta < 1$). The conditional distribution of equation (7) induces the conditional renormalization

$$\left(\mathcal{R}_p(F_D)\right)(\theta) = \frac{F_D(p\theta)}{F_D(p)} \quad (8)$$

($0 < p, \theta < 1$).¹

A CDF $F_D(\cdot)$ is thus a renormalization fixed point if and only if it satisfies the functional equation $F_D(xy) = F_D(x)F_D(y)$ ($0 < x, y < 1$). The solutions of this functional equation, in turn, are the *Beta* CDFs of equation (6).

For probability distributions supported on the unit interval $(0, 1)$ we conclude that:

- The CDF-fractal probability distributions are the *Beta distributions* of equation (6).
- The CDF-fractal renormalization is the *conditional renormalization* of equation (8).

¹In this case the renormalization parameter p is restricted to the range $0 < p < 1$.

2.1.2 SDF-fractality

The SDF $\overline{F}_D(\cdot)$ is monotone decreasing from the level $\lim_{\theta \rightarrow 0} \overline{F}_D(\theta) = 1$ to the level $\lim_{\theta \rightarrow \infty} \overline{F}_D(\theta) = 0$. Hence, in order that the SDF $\overline{F}_D(\cdot)$ admit a power-law structure its underlying probability distribution D must be *bounded from below*. Admissible power-law SDFs are thus of the form

$$\overline{F}_D(\theta) = \left(\frac{a}{\theta}\right)^\alpha \quad (9)$$

($\theta > a$), where the lower bound a and the exponent α are arbitrary positive parameters.

With no loss of generality, the lower bound can be set to unity ($a = 1$) – yielding the *Pareto* SDFs:

$$\overline{F}_D(\theta) = \theta^{-\alpha} \quad (10)$$

($\theta > 1$).

Let ξ denote a random variable drawn from an arbitrary probability distribution D supported on the ray $(1, \infty)$. The *conditional* distribution of the *scaled* random variable ξ/p – contingent on the information that the random variable ξ is *greater* than the level p – is given by

$$\mathbf{Prob}\left(\frac{\xi}{p} > \theta \mid \xi > p\right) = \frac{\mathbf{Prob}(\xi > p\theta)}{\mathbf{Prob}(\xi > p)} \quad (11)$$

($p, \theta > 1$). The conditional distribution of equation (11) induces the conditional renormalization

$$\left(\mathcal{R}_p(\overline{F}_D)\right)(\theta) = \frac{\overline{F}_D(p\theta)}{\overline{F}_D(p)} \quad (12)$$

($p, \theta > 1$).²

A SDF $\overline{F}_D(\cdot)$ is thus a renormalization fixed point if and only if it satisfies the functional equation $\overline{F}_D(xy) = \overline{F}_D(x)\overline{F}_D(y)$ ($x, y > 1$). The solutions of this functional equation, in turn, are the *Pareto* SDFs of equation (10).

For probability distributions supported on the ray $(1, \infty)$ we conclude that:

- The SDF-fractal probability distributions are the *Pareto distributions* of equation (10).
- The SDF-fractal renormalization is the *conditional renormalization* of equation (12).

²In this case the renormalization parameter p is restricted to the range $p > 1$.

2.1.3 Exponential representations

Both the aforementioned *Beta* and the *Pareto* probability distributions possess an underlying *Exponential structure* which we now describe.

Let ξ_{Beta} denote a random variable governed by the *Beta* CDF of equation (5); let ξ_{Pareto} denote a random variable governed by the *Pareto* SDF of equation (9); and, let \mathcal{E} denote an Exponentially-distributed random variable with unit mean. It is straightforward to observe that the following exponential representations holds:

$$\xi_{\text{Beta}} \stackrel{\text{Law}}{=} a \exp \left\{ -\frac{1}{\alpha} \mathcal{E} \right\} \quad \text{and} \quad \xi_{\text{Pareto}} \stackrel{\text{Law}}{=} a \exp \left\{ \frac{1}{\alpha} \mathcal{E} \right\} . \quad (13)$$

Note that equation (13) immediately implies a reciprocal connection between the *Beta* and the *Pareto* random variables:

$$\xi_{\text{Beta}} \stackrel{\text{Law}}{=} \frac{1}{\xi_{\text{Pareto}}} \quad \text{and} \quad \xi_{\text{Pareto}} \stackrel{\text{Law}}{=} \frac{1}{\xi_{\text{Beta}}} . \quad (14)$$

Let ξ denote a random variable drawn from an arbitrary positive-valued probability distribution D . The *conditional* distribution of the *translated* random variable $\xi - p$ – contingent on the information that the random variable ξ is *greater* than the level p – is given by

$$\mathbf{Prob} \left(\xi - p > \theta \mid \xi > p \right) = \frac{\mathbf{Prob} (\xi > p + \theta)}{\mathbf{Prob} (\xi > p)} \quad (15)$$

($p, \theta > 0$). The conditional distribution of equation (15) induces the conditional renormalization

$$\left(\mathcal{R}_p(\overline{F}_D) \right) (\theta) = \frac{\overline{F}_D(p + \theta)}{\overline{F}_D(p)} \quad (16)$$

($p, \theta > 1$).

Equations (15)-(16) are the *translational* counterparts of equations (7)-(8) and equations (11)-(12).

A SDF $\overline{F}_D(\cdot)$ is a renormalization fixed point of equation (16) if and only if it satisfies the functional equation $\overline{F}_D(x + y) = \overline{F}_D(x) \overline{F}_D(y)$ ($x, y > 0$). The unique unit-mean solution of this functional equation is the unit-mean *Exponential SDF*. This characterizing property of the *Exponential distribution* – often referred to as “lack of memory” ([12], Section XVII.6) – is of prime importance in probability theory and its applications. As we see here, this elemental property also underlies the CDF-fractal renormalization and the SDF-fractal renormalization.

2.2 Fractality via Hazard Rates

Let ξ denote a random variable drawn from an arbitrary positive-valued probability distribution D . What is the probability that the random variable ξ be

realized *at* the level θ – provided that it is *not* realized *above* the level θ ? The answer to this question is given by the Backward Hazard Rate (BHR) $H_D(\cdot)$, defined as follows:

$$H_D(\theta) = \lim_{\delta \rightarrow 0} \frac{1}{\delta} \mathbf{P} \left(\xi > \theta - \delta \mid \xi \leq \theta \right) = \frac{f_D(\theta)}{F_D(\theta)} \quad (17)$$

($\theta > 0$).

And what about the probability that the random variable ξ be realized *at* the level θ – provided that it is *not* realized *below* the level θ ? The answer to this analogous question is given by the Forward Hazard Rate (FHR) $\overline{H}_D(\cdot)$, defined as follows:

$$\overline{H}_D(\theta) = \lim_{\delta \rightarrow 0} \frac{1}{\delta} \mathbf{P} \left(\xi \leq \theta + \delta \mid \xi > \theta \right) = \frac{f(\theta)}{\overline{F}_D(\theta)} \quad (18)$$

($\theta > 0$). The FHR plays a central role in Applied Probability and in the Theory of Reliability [13]-[15].

Both the BHR and the FHR are distribution-characteristics. Indeed, the CDF and the SDF can be reconstructed, respectively, from the BHR and the FHR via

$$F_D(\theta) = \exp \left\{ - \int_{\theta}^{\infty} H_D(x) dx \right\} \quad (19)$$

($\theta > 0$), and via

$$\overline{F}_D(\theta) = \exp \left\{ - \int_0^{\theta} \overline{H}_D(x) dx \right\} \quad (20)$$

($\theta > 0$).

In this subsection we study BHR-fractality and FHR-fractality.

2.2.1 BHR-fractality

As indicated above, the CDF $F_D(\cdot)$ is monotone increasing from the level $\lim_{\theta \rightarrow 0} F_D(\theta) = 0$ to the level $\lim_{\theta \rightarrow \infty} F_D(\theta) = 1$. Hence, equation (19) implies that the BHR $H_D(\cdot)$ is integrable at infinity, and is non-integrable over the entire positive half-line ($\int_0^{\infty} H_D(x) dx = \infty$).

Admissible power-law BHRs thus yield the *Fréchet* CDFs

$$F_D(\theta) = \exp \left\{ -a\theta^{-\alpha} \right\} \quad (21)$$

($\theta > 0$), where the coefficient a and the exponent α are arbitrary positive parameters.

Let $\{\xi_1, \dots, \xi_n\}$ denote a sequence of n IID random variables drawn from an arbitrary positive-valued probability distribution D . The distribution of the *maximal* random variable $\max \{\xi_1, \dots, \xi_n\}$ – *scaled-down* by the multiplicative factor $n^{-1/\alpha}$ – is given by

$$\mathbf{Prob} \left(\frac{1}{n^{1/\alpha}} \max \{\xi_1, \dots, \xi_n\} \leq \theta \right) = \left(\mathbf{Prob} \left(\xi_1 \leq n^{1/\alpha} \theta \right) \right)^n \quad (22)$$

($\theta > 0$). The maximum distribution of equation (22) induces the maximal renormalization

$$\left(\mathcal{R}_p(F_D)\right)(\theta) = \left(F_D\left(p^{1/\alpha}\theta\right)\right)^p \quad (23)$$

($p, \theta > 0$).

A CDF $F_D(\cdot)$ is thus a renormalization fixed point if and only if its logarithm $G_D(\cdot) = \ln(F_D(\cdot))$ satisfies the functional equation $G_D(xy) = x^{-\alpha}G_D(y)$ ($x, y > 0$). The solutions of this functional equation, in turn, are the *Fréchet* CDFs of equation (21).

For positive-valued probability distributions we conclude that:

- The BHR-fractal probability distributions are the *Fréchet distributions* of equation (21).
- The BHR-fractal renormalization is the *maximal renormalization* of equation (23).

2.2.2 FHR-fractality

As indicated above, the SDF $\overline{F}_D(\cdot)$ is monotone decreasing from the level $\lim_{\theta \rightarrow 0} \overline{F}_D(\theta) = 1$ to the level $\lim_{\theta \rightarrow \infty} \overline{F}_D(\theta) = 0$. Hence, equation (20) implies that the FHR $\overline{H}_D(\cdot)$ is integrable at the origin, and is non-integrable over the entire positive half-line ($\int_0^\infty \overline{H}_D(x)dx = \infty$).

Admissible power-law FHRs thus yield the *Weibull* SDFs

$$\overline{F}_D(\theta) = \exp\{-a\theta^\alpha\} \quad (24)$$

($\theta > 0$), where the coefficient a and the exponent α are arbitrary positive parameters.

Let $\{\xi_1, \dots, \xi_n\}$ denote a sequence of n IID random variables drawn from an arbitrary positive-valued probability distribution D . The distribution of the *minimal* random variable $\min\{\xi_1, \dots, \xi_n\}$ – *scaled-up* by the multiplicative factor $n^{1/\alpha}$ – is given by

$$\mathbf{Prob}\left(n^{1/\alpha} \cdot \min\{\xi_1, \dots, \xi_n\} > \theta\right) = \left(\mathbf{Prob}\left(\xi_1 > \frac{\theta}{n^{1/\alpha}}\right)\right)^n \quad (25)$$

($\theta > 0$). The minimum distribution of equation (25) induces the minimal renormalization

$$\left(\mathcal{R}_p(\overline{F}_D)\right)(\theta) = \left(\overline{F}_D\left(\frac{\theta}{p^{1/\alpha}}\right)\right)^p \quad (26)$$

($p, \theta > 0$).

A SDF $\overline{F}_D(\cdot)$ is thus a renormalization fixed point if and only if its logarithm $\overline{G}_D(\cdot) = \ln(\overline{F}_D(\cdot))$ satisfies the functional equation $\overline{G}_D(xy) = x^\alpha \overline{G}_D(y)$ ($x, y >$

0). The solutions of the this functional equation, in turn, are the *Weibull* SDFs of equation (24).

For positive-valued probability distributions we conclude that:

- The FHR-fractal probability distributions are the *Weibull distributions* of equation (24).
- The FHR-fractal renormalization is the *minimal renormalization* of equation (26).

2.2.3 Exponential representations

Both the aforementioned *Fréchet* and the *Weibull* probability distributions posses an underlying *Exponential structure* which we now describe.

Let $\xi_{\text{Fréchet}}$ denote a random variable governed by the *Fréchet* CDF of equation (21); let ξ_{Weibull} denote a random variable governed by the *Weibull* SDF of equation (24); and, let \mathcal{E} denote an Exponentially-distributed random variable with unit mean. It is straightforward to observe that the following exponential representations hold:

$$\xi_{\text{Fréchet}} \stackrel{\text{Law}}{=} \left(\frac{1}{a}\mathcal{E}\right)^{1/\alpha} \quad \text{and} \quad \xi_{\text{Weibull}} \stackrel{\text{Law}}{=} \left(\frac{1}{a}\mathcal{E}\right)^{-1/\alpha}. \quad (27)$$

Note that equation (27) immediately implies a reciprocal connection between the *Fréchet* and the *Weibull* random variables:

$$\xi_{\text{Fréchet}} \stackrel{\text{Law}}{=} \frac{1}{\xi_{\text{Weibull}}} \quad \text{and} \quad \xi_{\text{Weibull}} \stackrel{\text{Law}}{=} \frac{1}{\xi_{\text{Fréchet}}}. \quad (28)$$

The *Exponential distribution* corresponds to the *minimal renormalization* of equation (26) with exponent $\alpha = 1$. Indeed, the unique unit-mean solution of this renormalization is the unit-mean *Exponential* SDF.

2.3 Fractality via Laplace-space characteristics

So forth, we considered fractality via “frequency-based” distribution-characteristics: CDFs, SDFs, BHRs, FHRs. In this Subsection we shift to *Laplace space* and turn to study fractality via the following “analytic-based” distribution-characteristics: Laplace Transforms; Moment Sequences; Log-Laplace Transforms; Cumulant Sequences.

2.3.1 Laplace Transforms and Moment Sequences

The Laplace Transform (LT) $L_D(\cdot)$ of a positive-valued probability distribution D is the Laplace Transform of its PDF $f_D(\cdot)$:

$$L_D(\theta) = \int_0^\infty \exp\{-\theta x\} f_D(x) dx \quad (29)$$

($\theta \geq 0$). The LT is a distribution-characteristic – though, in general, the reconstruction of a PDF from a given LT is hard a task [16].

The LT $L_D(\cdot)$ is monotone decreasing from the level $L_D(0) = 1$ to the level $\lim_{\theta \rightarrow \infty} L_D(\theta) = 0$. Hence, LTs *cannot* admit the power-law structure of equation (2).

In case the LT $L_D(\cdot)$ admits a Taylor expansion around the origin, the Moment Sequence (MS) $\{M_D(n)\}_{n=0}^{\infty}$ of the probability distribution D is well defined and is given by

$$L_D(\theta) = \sum_{n=0}^{\infty} M_D(n) \frac{(-\theta)^n}{n!} \quad (30)$$

($\theta \geq 0$). Reconstructing a probability distribution from a given MS is known as the *Stieltjes Moment Problem* [16].

Since the Moment of order zero equals unity ($M_D(0) = 1$) MSs *cannot* admit the power-law structure of equation (2).

We conclude that there are *no* LT-fractal and *no* MS-fractal probability distributions.

2.3.2 Log-Laplace Transforms

A “cousin” of the LT $L_D(\cdot)$ is its logarithm – referred to as the Log-Laplace Transform (LLT) $\Psi_D(\cdot)$ and given by

$$\Psi_D(\theta) = -\ln(L_D(\theta)) \quad (31)$$

($\theta \geq 0$).

The LLT initiates at the origin ($\Psi_D(0) = 0$) and is monotone increasing ($\Psi_D'(\theta) > 0$) and concave ($\Psi_D''(\theta) < 0$).

Admissible power-law LLTs are thus the *Lévy Stable* LLTs:

$$\Psi_D(\theta) = a\theta^\alpha \quad (32)$$

($\theta \geq 0$), where a is an arbitrary positive coefficient and where the exponent α takes values in the range $0 < \alpha < 1$. The *Lévy Stable* LLTs of equation (32) admit the integral representation

$$\Psi_D(\theta) = \theta \int_0^{\infty} \exp\{-\theta x\} \left(\frac{a}{\Gamma(1-\alpha)} \frac{1}{x^\alpha} \right) dx \quad (33)$$

($\theta \geq 0$) – whose meaning will be explained in the sequel.

(Apart from the special case $\alpha = 1/2$, there is no “closed form” representation for the PDFs of the *Lévy Stable* probability distributions.)

Let $\{\xi_1, \dots, \xi_n\}$ denote a sequence of n IID random variables drawn from an arbitrary positive-valued probability distribution D . The LT of the *aggregate* $\xi_1 + \dots + \xi_n$ – *scaled-down* by the multiplicative factor $n^{-1/\alpha}$ – is given by

$$\left\langle \exp \left\{ -\theta \frac{\xi_1 + \dots + \xi_n}{n^{1/\alpha}} \right\} \right\rangle = \left\langle \exp \left\{ -\frac{\theta}{n^{1/\alpha}} \xi_1 \right\} \right\rangle^n \quad (34)$$

($\theta \geq 0$). The LT of equation (34) induces the aggregative renormalization

$$\left(\mathcal{R}_p(\Psi_D) \right) (\theta) = p \Psi_D \left(\frac{\theta}{p^{1/\alpha}} \right) \quad (35)$$

($p > 0, \theta \geq 0$).

A LLT $\Psi_D(\cdot)$ is thus a renormalization fixed point if and only if it satisfies the functional equation $\Psi_D(xy) = x^{-\alpha} \Psi_D(y)$ ($x, y > 0$). The solutions of this functional equation, in turn, are the *Lévy Stable* LLTs of equation (32).

For positive-valued probability distributions we conclude that:

- The LLT-fractal probability distributions are the *Lévy Stable distributions* characterized by the LLTs of equation (32).
- The LLT-fractal renormalization is the *aggregative renormalization* of equation (35).

2.3.3 Cumulant Sequences

In case the LLT $\Psi_D(\cdot)$ admits a Taylor expansion around the origin, the Cumulant Sequence (CS) $\{C_D(n)\}_{n=1}^{\infty}$ of the probability distribution D is well defined and is given by

$$\Psi_D(\theta) = - \sum_{n=1}^{\infty} C_D(n) \frac{(-\theta)^n}{n!} \quad (36)$$

($\theta \geq 0$).

Power-law CSs of the form $C_D(n) = an^{-\alpha}$ ($n = 1, 2, \dots$), where the coefficient a and the exponent α are arbitrary positive parameters, yield LLTs admitting the following integral representation:

$$\Psi_D(\theta) = \theta \int_0^1 \exp \{ -\theta x \} \left(\frac{a}{\Gamma(1+\alpha)} (-\ln(x))^\alpha \right) dx \quad (37)$$

($\theta \geq 0$).

The proof of equation (37) is given in the Appendix; the meaning of this integral representation will be explained in the sequel. The renormalization associated with CS-fractal probability distributions is based on their underlying Poissonian structure – which, too, will be explained in the sequel.

For positive-valued probability distributions we conclude that:

- The CS-fractal probability distributions are characterized by the LLTs of equation (37).
- The CS-fractal renormalization is a *Poissonian renormalization* (yet to be presented).

2.4 Interim summery

Table 1 summarizes the six classes of fractal probability distributions characterized in this Section.

We note that in the context of IID sequences of positive-valued random variables: **(i)** Extreme Value Theory asserts that the *Fréchet* and *Weibull* distributions are, respectively, the only possible linear scaling limits of the sequences' maxima and minima [17]-[19]; **(ii)** the Central Limit Theorem asserts that the one-sided *Lévy Stable* distribution is the only possible linear scaling limit of the sequences' sums [20]-[22].

Table 1
The classes of fractal probability distributions

Fractality	Distribution	Renormalization
CDF-fractal	<i>Beta</i>	Conditional
SDF-fractal	<i>Pareto</i>	Conditional
BHR-fractal	<i>Fréchet</i>	Maximal
FHR-fractal	<i>Weibull</i>	Minimal
LLT-fractal	<i>Lévy Stable</i>	Aggregative
CS-fractal	—	Poissonian

3 Poissonian representation of fractal distributions

In this section we provide *Poissonian representations* for all six classes of fractal probability distributions characterized in the previous Section.

3.1 Poisson processes

In this Subsection we recall the notion of Poisson processes. For further details the readers are referred to [23].

A *Poisson process* \mathcal{X} on the positive half-line, with *rate function* $r(\cdot)$, is a random collection of positive-valued points satisfying the following properties: **(i)** the number of points $N_{\mathcal{X}}(I)$ residing in the interval I is a Poisson-distributed random variable with mean $\int_I r(x)dx$; and, **(ii)** if $\{I_k\}_k$ is a finite collection of disjoint intervals then $\{N_{\mathcal{X}}(I_k)\}_k$ is a finite collection of independent random variables.

The rate function $r(\cdot)$ is the Poissonian analogue of the PDF in the context of probability distributions. The Poissonian analogues of the CDF and the SDF, respectively, are: **(i)** the Cumulative Rate Function (CRF) $R(\cdot)$, given by

$$R(\theta) = \int_0^\theta r(x)dx \quad (38)$$

($\theta > 0$); and, **(ii)** the Survival Rate Function (SRF) $\overline{R}(\cdot)$, given by

$$\overline{R}(\theta) = \int_\theta^\infty r(x)dx \quad (39)$$

($\theta > 0$).

The CRF $R(\cdot)$ is well defined if and only if the rate function $r(\cdot)$ is integrable at the origin – in which case it is a monotone non-decreasing function initiating from the origin ($R(0) = 0$). The SRF $\overline{R}(\cdot)$ is well defined if and only if the rate function $r(\cdot)$ is integrable at infinity – in which case it is a monotone non-increasing function decreasing to zero ($\lim_{\theta \rightarrow \infty} \overline{R}(\theta) = 0$).

The *average* number of points of the Poisson process \mathcal{X} residing below the level θ is given by the CRF value $R(\theta)$; the *average* number of points residing above the level θ is given by the SRF value $\overline{R}(\theta)$.

3.2 Poissonian maxima

Consider the *maximum* of the Poisson process \mathcal{X} , defined as follows:

$$X_{\max} = \max_{x \in \mathcal{X}} \{x\} . \quad (40)$$

We refer to the probability distribution of the random variable X_{\max} as the *maximal distribution* of the Poisson process \mathcal{X} .

The maximum X_{\max} is smaller than the level θ ($\theta > 0$) if and only if the process \mathcal{X} has no points residing above this level. Namely: $\{X_{\max} \leq \theta\} = \{N_{\mathcal{X}}((\theta, \infty)) = 0\}$. Since the random variable $N_{\mathcal{X}}((\theta, \infty))$ is Poisson-distributed with mean $\overline{R}(\theta)$, we obtain that the maximal distribution of the Poisson process \mathcal{X} is characterized by the CDF

$$F_{\max}(\theta) = \exp\{-\overline{R}(\theta)\} \quad (41)$$

($\theta > 0$).

Equation (41) implies a one-to-one correspondence between Poisson processes (characterized by their SRFs $\overline{R}(\cdot)$) and their associated maximal distributions (characterized by their CDFs $F_{\max}(\cdot)$). This one-to-one correspondence yields the following Poissonian representation of the CDF-fractal and BHR-fractal probability distributions:

- A probability distribution D is *CDF-fractal* if and only if it is the maximal distribution of a Poisson process \mathcal{X} with *logarithmic* SRF of the form

$$\overline{R}(\theta) = -\ln\left(\left(\frac{\theta}{a}\right)^{\alpha}\right) \quad (42)$$

($0 < \theta < a$), where the upper bound a and the exponent α are arbitrary positive parameters.

- A probability distribution D is *BHR-fractal* if and only if it is the maximal distribution of a Poisson process \mathcal{X} with *power-law* SRF of the form

$$\overline{R}(\theta) = a\theta^{-\alpha} \quad (43)$$

($\theta > 0$), where the coefficient a and the exponent α are arbitrary positive parameters.

Equations (42) and (43) follow, respectively, from equations (5) and (21).

3.3 Poissonian minima

Consider the *minimum* of the Poisson process \mathcal{X} , defined as follows:

$$X_{\min} = \min_{x \in \mathcal{X}} \{x\} . \quad (44)$$

We refer to the probability distribution of the random variable X_{\min} as the *minimal distribution* of the Poisson process \mathcal{X} .

The minimum X_{\min} is larger than the level θ ($\theta > 0$) if and only if Poisson process \mathcal{X} has no points residing below this level. Namely: $\{X_{\min} > \theta\} = \{N_{\mathcal{X}}((0, \theta]) = 0\}$. Since the random variable $N_{\mathcal{X}}((0, \theta])$ is Poisson-distributed with mean $R(\theta)$, we obtain that the minimal distribution of the Poisson process \mathcal{X} is characterized by the CDF

$$\overline{F}_{\min}(\theta) = \exp\{-R(\theta)\} \quad (45)$$

($\theta > 0$).

Equation (45) implies a one-to-one correspondence between Poisson processes (characterized by their CRFs $R(\cdot)$) and their associated minimal distributions (characterized by their SDFs $\bar{F}_{\min}(\cdot)$). This one-to-one correspondence yields the following Poissonian representation of the SDF-fractal and FHR-fractal probability distributions:

- A probability distribution D is *SDF-fractal* if and only if it is the minimal distribution of a Poisson process \mathcal{X} with *logarithmic* CRF of the form

$$R(\theta) = -\ln \left(\left(\frac{a}{\theta} \right)^\alpha \right) \quad (46)$$

($\theta > a$), where the lower bound a and the exponent α are arbitrary positive parameters.

- A probability distribution D is *FHR-fractal* if and only if it is the minimal distribution of a Poisson process \mathcal{X} with *power-law* CRF of the form

$$R(\theta) = a\theta^\alpha \quad (47)$$

($\theta > 0$), where the coefficient a and the exponent α are arbitrary positive parameters.

Equations (46) and (47) follow, respectively, from equations (9) and (24).

3.4 Poissonian aggregates

Consider the *aggregate* of the Poisson process \mathcal{X} , defined as follows:

$$X_{\text{agg}} = \sum_{x \in \mathcal{X}} x. \quad (48)$$

We refer to the probability distribution of the random variable X_{agg} as the *aggregate distribution* of the Poisson process \mathcal{X} .

The aggregate of equation (48) can be either convergent ($X_{\text{agg}} < \infty$) or divergent ($X_{\text{agg}} = \infty$). Campbell's theorem of the theory of Poisson processes ([23], Section 3.2) implies that the aggregate is convergent if and only if the SRF $\bar{R}(\cdot)$ is integrable at the origin – in which case the aggregate distribution of the Poisson process \mathcal{X} is characterized by the LLT

$$\Psi_{\text{agg}}(\theta) = \theta \int_0^\infty \exp\{-\theta x\} \bar{R}(x) dx \quad (49)$$

($\theta \geq 0$).

Equation (49) implies a one-to-one correspondence between Poisson processes (characterized by their SRFs $\bar{R}(\cdot)$) and their associated aggregate distributions (characterized by their LLTs $\Psi_{\text{agg}}(\cdot)$) – which, in turn, are characterized by the Laplace transforms of the underlying SRFs). This one-to-one correspondence yields the following Poissonian representation of the LLT-fractal and CS-fractal probability distributions:

- A probability distribution D is *LLT-fractal* if and only if it is the aggregate distribution of a Poisson process \mathcal{X} with *power-law* SRF of the form

$$\overline{R}(\theta) = \frac{a}{\Gamma(1-\alpha)} \frac{1}{\theta^\alpha} \quad (50)$$

($\theta > 0$), where a is an arbitrary positive coefficient and where the exponent α takes values in the range $0 < \alpha < 1$.

- A probability distribution D is *CS-fractal* if and only if it is the aggregate distribution of a Poisson process \mathcal{X} with *logarithmic* SRF of the form

$$\overline{R}(\theta) = \frac{a}{\Gamma(1+\alpha)} \left(-\ln(\theta) \right)^\alpha \quad (51)$$

($0 < \theta < 1$), where the coefficient a and the exponent α are arbitrary positive parameters.

Equations (50) and (51) follow, respectively, from equations (33) and (37).

4 The underlying Poissonian fractals

In Section 2 we characterized six classes of fractal probability distributions – each stemming from a different distribution-characteristic, and each associated with a different renormalization. In section 3 we have further seen that all six classes of fractal probability distributions admit Poissonian representations – either maximal, minimal, or aggregative.

Is there any kind of an underlying order to this “little zoo” of fractal distributions?

The answer is affirmative: all fractal distributions obtained are *functional projections* of underlying *Poissonian fractals* – as we shall show in this Section.

4.1 Poissonian renormalizations and their fixed points

In this Subsection we study renormalizations of Poisson processes defined on the positive half-line. We follow the renormalization approach used in [11].

4.1.1 Poissonian renormalizations

Let $\{\phi_p\}_{p>0}$ be a family of *consistent scaling functions*: monotone-increasing functions which map the positive half-line $(0, \infty)$ onto itself, and which satisfy the “consistency condition” $\phi_p \circ \phi_q = \phi_{pq}$ ($p, q > 0$; the sign \circ denoting composition).

Given a Poisson process \mathcal{X} with rate function $r(\cdot)$ we construct its *p-order renormalization* \mathcal{X}_p via the following two-step algorithm: **(i)** replace the process \mathcal{X} by an intermediate Poisson process $\mathcal{X}_p^{\text{int}}$ with rate function $r_p^{\text{int}}(\cdot) = p \cdot$

$r(\cdot)$;³ (ii) shift the points of the intermediate process $\mathcal{X}_p^{\text{int}}$ using the p^{th} scaling function ϕ_p . The resulting p -order renormalization is given by

$$\mathcal{X}_p = \{\phi_p(x)\}_{x \in \mathcal{X}_p^{\text{int}}} \quad .$$

(The “consistency condition” is required in order to ensure that the Poissonian renormalization is consistent. Namely, that a p -order renormalization followed by a q -order renormalization equals a pq -order renormalization.)

The connection between the CRF $R_p(\cdot)$ and the SRF $\bar{R}_p(\cdot)$ of the p -order renormalization \mathcal{X}_p , and the CRF $R(\cdot)$ and the SRF $\bar{R}(\cdot)$ of the original process \mathcal{X} , is given by: $R_p(\cdot) = pR(\phi_p^{-1}(\cdot))$ and $\bar{R}_p(\cdot) = p\bar{R}(\phi_p^{-1}(\cdot))$, where the function $\phi_p^{-1}(\cdot)$ denotes the inverse of the scaling function $\phi_p(\cdot)$ (these results are an immediate consequence of the “displacement theorem” of the theory of Poisson processes – see Section 5.5 in [23]).

Denoting by $\mathcal{R} = \{\mathcal{R}_p\}_{p>0}$ the Poissonian renormalization defined, we have:

$$\mathcal{R}_p(R) = p(R \circ \phi_p^{-1}) \quad \text{and} \quad \mathcal{R}_p(\bar{R}) = p(\bar{R} \circ \phi_p^{-1}) \quad (52)$$

($p > 0$).

A Poisson process \mathcal{X} is a *fixed point* of the renormalization \mathcal{R} if it is left statistically unchanged by the renormalization’s action: the p -order renormalization \mathcal{X}_p being equal, in law, to the original process \mathcal{X} . In terms of the CRF and SRF it is required that $\mathcal{R}_p(R) = R$ and $\mathcal{R}_p(\bar{R}) = \bar{R}$ (for all $p > 0$). Using equation (52) we conclude that: the Poisson process \mathcal{X} is a renormalization fixed point if and only if its CRF $R(\cdot)$ and SRF $\bar{R}(\cdot)$ satisfy

$$R \circ \phi_p = pR \quad \text{and} \quad \bar{R} \circ \phi_p = p\bar{R} \quad (53)$$

($p > 0$).

The two most fundamental Poissonian renormalizations are *multiplicative* and *power-law*. We now turn to characterize the fixed points of these renormalizations.

4.1.2 Fixed points of multiplicative renormalizations

A *multiplicative renormalization* is based on a set of multiplicative scaling functions $\{\phi_p\}_{p>0}$. The consistency condition implies that the multiplicative scaling functions admit the form

$$\phi_p(x) = p^\varepsilon x \quad (54)$$

($x > 0$), where the exponent ε is an arbitrary non-zero parameter.

The fixed points of a multiplicative renormalization governed by the scaling functions of equation (54) are as follows (the coefficient c being an arbitrary positive parameter):

³Note that if p is integer then the intermediate Poisson process $\mathcal{X}_p^{\text{int}}$ is the *union* of p IID copies of the original Poisson process \mathcal{X} .

- In the case of a *positive* exponent ε the renormalization fixed points are characterized by the CRF

$$R(\theta) = c\theta^{1/\varepsilon} \quad (\theta > 0). \quad (55)$$

The accumulation point of these fixed-point Poisson processes is $x_* = \infty$.⁴

- In the case of a *negative* exponent ε the renormalization fixed points are characterized by the SRF

$$\overline{R}(\theta) = c\theta^{1/\varepsilon} \quad (\theta > 0). \quad (56)$$

The accumulation point of these fixed-point Poisson processes is $x_* = 0$.

4.1.3 Fixed points of power-law renormalizations

A *power-law renormalization* is based on a set of power-law scaling functions $\{\phi_p\}_{p>0}$. The consistency condition implies that the power-law scaling functions admit the form

$$\phi_p(x) = x^{p^\varepsilon} \quad (57)$$

($x > 0$), where the exponent ε is an arbitrary non-zero parameter.

The fixed points of a power-law renormalization governed by the scaling functions of equation (57) cannot range over the entire positive half-line $(0, \infty)$. Rather, they may range either on the unit interval $(0, 1)$ or on the ray $(1, \infty)$ (note that the power-law scaling functions of equation (57) indeed map the unit interval $(0, 1)$ and the ray $(1, \infty)$, respectively, onto themselves).

The fixed points of a power-law renormalization governed by the scaling functions of equation (57), and ranging over the unit interval $(0, 1)$, are as follows (the coefficient c being an arbitrary positive parameter):

- In the case of a *positive* exponent ε the renormalization fixed points are characterized by the SRF

$$\overline{R}(\theta) = c(-\ln \theta)^{1/\varepsilon} \quad (0 < \theta < 1). \quad (58)$$

The accumulation point of these fixed-point Poisson processes is $x_* = 0$.

- In the case of a *negative* exponent ε the renormalization fixed points are characterized by the CRF

$$R(\theta) = c(-\ln \theta)^{1/\varepsilon} \quad (0 < \theta < 1). \quad (59)$$

The accumulation point of these fixed-point Poisson processes is $x_* = 1$.

⁴A point x_* ($0 \leq x_* \leq \infty$) is said to be an *accumulation point* of the Poisson process \mathcal{X} if, with probability one, there are infinitely many points of \mathcal{X} within any given neighborhood of x_* .

The fixed points of a power-law renormalization governed by the scaling functions of equation (57), and ranging over the ray $(1, \infty)$, are as follows (the coefficient c being an arbitrary positive parameter):

- In the case of a *positive* exponent ε the renormalization fixed points are characterized by the CRF

$$R(\theta) = c(\ln \theta)^{1/\varepsilon} \quad (\theta > 1). \quad (60)$$

The accumulation point of these fixed-point Poisson processes is $x_* = \infty$.

- In the case of a *negative* exponent ε the renormalization fixed points are characterized by the SRF

$$\overline{R}(\theta) = c(\ln \theta)^{1/\varepsilon} \quad (\theta > 1). \quad (61)$$

The accumulation point of these fixed-point Poisson processes is $x_* = 1$.

4.2 Poissonian fractals

In the previous Subsection we obtained six classes of renormalization fixed-point Poisson processes. Excluding the fixed-point processes whose accumulation point is $x_* = 1$ (an interior point of the positive half-line), and considering the fixed-point processes whose accumulation point is either the origin $x_* = 0$ or infinity $x_* = \infty$ (the boundaries of the positive half-line), we now define four classes of *Poissonian fractals*. The fractal distributions of Section 2 shall turn out to be one-dimensional functional projections – either maximal, minimal, or aggregative – of these underlying Poissonian fractals.

4.2.1 Linear Poissonian fractals

The class of *linear Poissonian fractals* comprises of all Poisson processes governed by CRFs admitting the power-law form

$$R(\theta) = a\theta^\alpha \quad (\theta > 0), \quad (62)$$

where the coefficient a and the exponent α are arbitrary positive parameters. The members of this class are fixed points of *multiplicative* Poissonian renormalizations. For this class:

- The maximal distribution is degenerate: the maximum X_{\max} equals infinity with probability one.
- The minimal distribution is the FHR-fractal *Weibull distribution*, characterized by the SDF

$$\overline{F}_{\min}(\theta) = \exp \{-a\theta^\alpha\} \quad (63)$$

$(\theta > 0)$.

- The aggregate distribution is degenerate: the aggregate X_{agg} is infinite with probability one.

4.2.2 Harmonic Poissonian fractals

The class of *harmonic Poissonian fractals* comprises of all Poisson processes governed by SRFs admitting the power-law form

$$\overline{R}(\theta) = a\theta^{-\alpha} \quad (\theta > 0), \quad (64)$$

where the coefficient a and the exponent α are arbitrary positive parameters. The members of this class are fixed points of *multiplicative* Poissonian renormalizations. For this class:

- The maximal distribution is the BHR-fractal *Fréchet distribution*, characterized by the CDF

$$F_{\max}(\theta) = \exp \{-a\theta^{-\alpha}\} \quad (65)$$

$(\theta > 0)$.

- The minimal distribution is degenerate: the minimum X_{\min} equals zero with probability one.
- If the exponent α is in the range $0 < \alpha < 1$ then the aggregate distribution is the LLT-fractal *Lévy Stable distribution*, characterized by the LLT

$$\Psi_{\text{agg}}(\theta) = \Gamma(1 - \alpha)a\theta^\alpha \quad (66)$$

$(\theta > 0)$.

- If the exponent α is in the range $\alpha \geq 1$ then the aggregate distribution is degenerate: the aggregate X_{agg} is infinite with probability one.

4.2.3 Log-linear Poissonian fractals

The class of *log-linear Poissonian fractals* comprises of all Poisson processes governed by CRFs admitting the logarithmic form

$$R(\theta) = a(\ln(\theta))^\alpha \quad (\theta > 1), \quad (67)$$

where the coefficient a and the exponent α are arbitrary positive parameters. The members of this class are fixed points of *power-law* Poissonian renormalizations. For this class:

- The maximal distribution is degenerate: the maximum X_{\max} equals infinity with probability one.
- The minimal distribution is characterized by the SDF

$$\overline{F}_{\min}(\theta) = \exp \{-a(\ln(\theta))^\alpha\} \quad (68)$$

$(\theta > 1)$.

- The aggregate distribution is degenerate: the aggregate X_{agg} is infinite with probability one.

The SDF of equation (68) reduces to the *Pareto* SDF when setting the exponent value α to unity. Thus, we refer to the distribution corresponding to this SDF as *Hyper Pareto*.

4.2.4 Log-harmonic Poissonian fractals

The class of *log-harmonic Poissonian fractals* comprises of all Poisson processes governed by SRFs admitting the logarithmic form

$$\overline{R}(\theta) = a (-\ln(\theta))^\alpha \quad (0 < \theta < 1), \quad (69)$$

where the coefficient a and the exponent α are arbitrary positive parameters. The members of this class are fixed points of *power-law* Poissonian renormalizations. For this class:

- The maximal distribution is characterized by the CDF

$$F_{\max}(\theta) = \exp \{-a (-\ln(\theta))^\alpha\} \quad (70)$$

$(0 < \theta < 1)$.

- The minimal distribution is degenerate: the minimum X_{\min} equals zero with probability one.
- The aggregate distribution is the CS-fractal distribution, characterized by the CS

$$C_{\text{agg}}(n) = \frac{\Gamma(1 + \alpha)a}{n^\alpha} \quad (71)$$

$(n = 1, 2, \dots)$.

The CDF of equation (70) reduces to the *Beta* CDF when setting the exponent value α to unity. Thus, we refer to the distribution corresponding to this CDF as *Hyper Beta*.

4.3 Structural properties of Poissonian fractals

In this Subsection we describe the structural properties of the four classes of Poissonian fractals presented in the previous Subsection.

4.3.1 Power-law structure and intrinsic scales

The CRFs of the linear and log-linear Poissonian fractals admit the power-law structure

$$R(\cdot) = a \left(S(\cdot) \right)^\alpha, \quad (72)$$

where the coefficient a and the exponent α are arbitrary positive parameters, and where the function $S(\cdot)$ is the *intrinsic scale* of the class under consideration:

- *Linear scale* $S(\theta) = \theta$ in the case of linear Poissonian fractals ($\theta > 0$).
- *Log-linear scale* $S(\theta) = \ln(\theta)$ in the case of log-linear Poissonian fractals ($\theta > 1$).

Analogously, the SRFs of harmonic and log-harmonic Poissonian fractals admit the power-law structure

$$\overline{R}(\cdot) = a \left(S(\cdot) \right)^\alpha, \quad (73)$$

where the coefficient a and the exponent α are arbitrary positive parameters, and where the function $S(\cdot)$ is the *intrinsic scale* of the class under consideration:

- *Harmonic scale* $S(\theta) = \theta^{-1}$ in the case of harmonic Poissonian fractals ($\theta > 0$).
- *Log-harmonic scale* $S(\theta) = \ln(\theta^{-1})$ in the case of log-harmonic Poissonian fractals ($0 < \theta < 1$).

All four classes of Poissonian fractals share the *common* power-law structure $y = ax^\alpha$. What distinguishes one fractal class from another is the *intrinsic scale* – on which the power-law structure is composed.

4.3.2 Order statistics and Exponential representations

All four classes of Poissonian fractals possess an underlying *Exponential structure* which we now describe. Let $\{\mathcal{E}_n\}_{n=1}^\infty$ denote an IID sequence of Exponentially-distributed random variables with unit mean.

The points of linear and log-linear Poissonian fractals can be listed in an increasing order $\xi_1 < \xi_2 < \xi_3 < \dots$. The *order statistics* $\{\xi_n\}_{n=1}^\infty$, in turn, admit the following exponential representations:

- Linear Poissonian fractals:

$$\xi_n \stackrel{\text{Law}}{=} \left(\frac{\mathcal{E}_1 + \dots + \mathcal{E}_n}{a} \right)^{1/\alpha} \quad (74)$$

($n = 1, 2, \dots$). Equation (74) is the infinite-dimensional counterpart of the one-dimensional exponential representations of the Fréchet distribution given in equation (27).

- Log-linear Poissonian fractals:

$$\xi_n \stackrel{\text{Law}}{=} \exp \left\{ \left(\frac{\mathcal{E}_1 + \dots + \mathcal{E}_n}{a} \right)^{1/\alpha} \right\} \quad (75)$$

($n = 1, 2, \dots$). Equation (75) is the infinite-dimensional generalization of the one-dimensional exponential representations of the Pareto distribution given in equation (13).

Analogously, the points of harmonic and log-harmonic Poissonian fractals can be listed in a decreasing order $\xi_1 > \xi_2 > \xi_3 > \dots$. The *order statistics* $\{\xi_n\}_{n=1}^\infty$, in turn, admit the following exponential representations:

- Harmonic Poissonian fractals:

$$\xi_n \stackrel{\text{Law}}{=} \left(\frac{\mathcal{E}_1 + \cdots + \mathcal{E}_n}{a} \right)^{-1/\alpha} \quad (76)$$

($n = 1, 2, \dots$). Equation (76) is the infinite-dimensional counterpart of the one-dimensional exponential representations of the Weibull distribution given in equation (27).

- Log-harmonic Poissonian fractals:

$$\xi_n \stackrel{\text{Law}}{=} \exp \left\{ - \left(\frac{\mathcal{E}_1 + \cdots + \mathcal{E}_n}{a} \right)^{1/\alpha} \right\} \quad (77)$$

($n = 1, 2, \dots$). Equation (77) is the infinite-dimensional generalization of the one-dimensional exponential representations of the Beta distribution given in equation (13).

Equations (74)-(77) follow from the “displacement theorem” of the theory of Poisson processes ([23], Section 5.5), combined with the fact that the increasing sequence $\{\mathcal{E}_1 + \cdots + \mathcal{E}_n\}_{n=1}^\infty$ forms a standard unit-rate Poisson process.

The reciprocal connection between equations (74) and (76) is the infinite-dimensional counterpart of the one-dimensional reciprocal connection between the Fréchet and Weibull distributions given by equation (28); the reciprocal connection between equations (75) and (77) is the infinite-dimensional generalization of the one-dimensional reciprocal connection between the Beta and Pareto distributions given by equation (14).

4.3.3 Transforming between fractal classes

It is possible to transform from an “input” Poissonian fractal \mathcal{X} belonging to one fractal class to an “output” Poissonian fractal \mathcal{Y} belonging to another fractal class via a simple point-to-point mapping $x \mapsto y = \psi(x)$ – which transforms the points x of the “input” Poissonian fractal to the points y of the “output” Poissonian fractal.

The point-to-point mappings are given in Table 2, which should be read as follows: in order to transform from the fractal class of row i to the fractal class of column j one has to apply the point-to-point mapping $y = \psi(x)$ appearing in cell (i, j) of the Table. The construction of these point-to-point mappings follows straightforwardly from the “displacement theorem” of the theory of Poisson processes ([23], Section 5.5).

Each class of Poissonian fractals has one degenerate extremal and one non-degenerate extremal – see Table 3 below. The point-to-point mappings of Table 2 transform the degenerate extremals amongst themselves, and transform the non-degenerate extremals amongst themselves. The point-to-point mappings of Table 2 do *not*, however, transform the aggregates of the different Poissonian fractal classes to each other.

Table 2
Point-to-point mappings of the Poissonian fractal classes

	Lin.	Har.	Log-lin.	Log-har.
Lin.	x	x^{-1}	$\exp(x)$	$\exp(-x)$
Har.	x^{-1}	x	$\exp(x^{-1})$	$\exp(-x^{-1})$
Log-lin.	$\ln(x)$	$(\ln(x))^{-1}$	x	x^{-1}
Log-har.	$-\ln(x)$	$-(\ln(x))^{-1}$	x^{-1}	x

Table 3
Extremals of the Poissonian fractal classes

Fractal Class	Degenerate extremals	Non-degenerate extremals
Linear	Max = ∞	Min \sim <i>Weibull</i>
Harmonic	Min = 0	Max \sim <i>Fréchet</i>
Log-linear	Max = ∞	Min \sim <i>Hyper Pareto</i>
Log-harmonic	Min = 0	Max \sim <i>Hyper Beta</i>

5 Conclusions

In this paper we examined the definition of fractality in the context of positive-valued probability distributions. We followed the conventional approach of associating the notion of fractality with power-law structures – considering various distribution characteristics, rather than the survival probability alone.

We proved the existence of no less than six different classes of fractal probability distributions – all admitting a characteristic power-law structure, and all being the unique fixed-points of renormalizations acting on positive-valued probability distributions. Each class manifested a markedly different meaning of fractality.

All fractal classes were further shown to admit an underlying Poissonian structure – each fractal distribution being a one-dimensional functional projection of an underlying Poisson process. The underlying Poisson processes, in turn, are fractal objects – being the unique fixed-points of Poissonian renormalizations.

The notion of fractality on the one-dimensional “probability-distribution level” emanated from the notion of fractality on the infinite-dimensional “Poisson-process level”:

On the “probability-distribution level” fractality was defined algebraically via power-law structures, and the connection between the different classes of fractal distributions was unclear.

On the elemental “Poisson-process level”, however, fractality was defined via the geometric notion of population-renormalization, and a unified picture of fractality was obtained: it became vividly clear how all classes of fractal distributions emerge from the underlying Poissonian fractals, how they connect to each other, and how the underlying Poissonian fractals connect to each other.

We have seen that on the elemental “Poisson-process level”, fractals do admit a universal power-law structure, yet this structure is intertwined with another key structure: the intrinsic scale which can be either linear, harmonic, log-linear, or log-harmonic. Whereas the power-law structure is common to all Poissonian fractals, it is the intrinsic scale which differentiates between the Poissonian fractal classes and characterizes them.

This research provides a panoramic and comprehensive view of fractal distributions, backed by a unified theory of their underlying Poissonian fractals.

6 Appendix: proof of equation (37)

We compute the LLT $\Psi_D(\theta)$ ($\theta \geq 0$) corresponding to the power-law CS $C_D(n) = an^{-\alpha}$ ($n = 1, 2, \dots$).

Note that

$$C_D(n) = \frac{a}{n^\alpha} = a \int_0^\infty \exp\{-nt\} \frac{t^{\alpha-1}}{\Gamma(\alpha)} dt \quad (78)$$

($n = 1, 2, \dots$). Hence, substituting equation (78) into equation (36) gives

$$\begin{aligned} \Psi_D(\theta) &= - \sum_{n=1}^\infty C_D(n) \frac{(-\theta)^n}{n!} \\ &= - \sum_{n=1}^\infty \left(a \int_0^\infty \exp\{-nt\} \frac{t^{\alpha-1}}{\Gamma(\alpha)} dt \right) \frac{(-\theta)^n}{n!} \\ &= - \int_0^\infty \left(\sum_{n=1}^\infty \frac{(-\theta \exp\{-t\})^n}{n!} \right) \left(\frac{a}{\Gamma(\alpha)} t^{\alpha-1} \right) dt \\ &= \int_0^\infty \left(1 - \exp\{-\theta \exp\{-t\}\} \right) \left(\frac{a}{\Gamma(\alpha)} t^{\alpha-1} \right) dt \end{aligned} \quad (79)$$

($\theta \geq 0$).

Now:

$$\Psi_D(\theta) = \int_0^\infty \left(1 - \exp\{-\theta \exp\{-t\}\} \right) \left(\frac{a}{\Gamma(\alpha)} t^{\alpha-1} \right) dt \quad (80)$$

(using the change of variables $u = \exp\{-t\}$)

$$= \int_0^1 \left(1 - \exp\{-\theta u\} \right) \left(\frac{a}{\Gamma(\alpha)} \frac{(-\ln(u))^{\alpha-1}}{u} \right) du \quad (81)$$

(using integration by parts)

$$= \int_0^1 \theta \exp\{-\theta x\} \left(\int_x^1 \frac{a}{\Gamma(\alpha)} \frac{(-\ln(u))^{\alpha-1}}{u} du \right) dx. \quad (82)$$

On the other hand (using the change of variables $t = -\ln(u)$) we have

$$\int_x^1 \frac{a}{\Gamma(\alpha)} \frac{(-\ln(u))^{\alpha-1}}{u} du = \int_0^{-\ln(x)} \frac{a}{\Gamma(\alpha)} t^{\alpha-1} dt = \frac{a}{\Gamma(1+\alpha)} (-\ln(x))^\alpha. \quad (83)$$

Hence, substituting equation (83) into equation (82) we conclude that

$$\Psi_D(\theta) = \theta \int_0^1 \exp\{-\theta x\} \left(\frac{a}{\Gamma(1+\alpha)} (-\ln(x))^\alpha \right) dx \quad (84)$$

($\theta \geq 0$).

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